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GENUS OF CURVES IN SMOOTH HYPERSURFACES

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Abstract. This is the continuation of our paper [9]. In this paper which is self contained, we would like to give a different obstruction formula to the FIRST order deformation of the pair of a smooth curve and a smooth hypersurface. This obstruction formula leads to a genus formula for a smooth curve in a smooth hypersurface. As an application we show that smooth elliptic curves in a smooth hypersurface of degree

$$h \geq 2n-1$$

in the projective space $\mathbf{P}^n, n \geq 3$, can't deform in the first order to all hypersurfaces of the same degree. In particular, there are no smooth elliptic curves in generic hypersurfaces of degree

$$h \geq 2n-1.$$

This application in return leads to a study of the deformation of the pair mentioned above.

1. Introduction. Our study of the genus of a curve is originated from our study of the obstructions to deformation of pairs of varieties. We hope the numerical bounds and invariants we obtained can support the general study of the deformation of pairs of varieties.

Let's briefly introduce the deformation in consideration. Let $f_0 \subset \mathbf{P}^n$ be a smooth hypersurface of degree h, where $\mathbf{P}^n$ is the projective space of dimension $n \geq 3$ over the complex numbers. We'll denote the section in $H^0(\mathcal{O}_{\mathbf{P}^n}(h))$ that defines f_0 also by f_0. Let $C_0 \subset f_0$ be a smooth curve. We investigate the existence of a family of pairs $C_t \subset f_t$, the curves C_t of degree d and the hypersurfaces f_t of degree h in the projective space $\mathbf{P}^n$ where t is in a variety. We call $C_t \subset f_t$, a "full" deformation of the pair. If C_t, f_t are algebraic and $\{f_t\}_{all\ t}$ form an open set of the space of all hypersurfaces, then around C_0, f_0, they can be trimmed to a versal subvariety defined by Clemens and Ran in [3]. A similar question was also investigated by L. Chiantini, Z. Ran in [4]. In general there is a well-known Kodaira's deformation theory ([6]) about the fibred submanifold $C_0 \subset f_0$ in a fibred complex manifold f_0, that says a sufficient condition for the C_0 to deform to all the other submanifolds is

$$H^1(N_{C_0} f_0) = 0.$$

But in general, it is not clear that this condition is also a necessary condition, i.e if $H^1(N_{C_0} f_0) \neq 0$, C_0 may still be able to deform to all the other hypersurfaces (We don't have a proof of that yet). In general, it may seem to be obvious that $H^1(N_{C_0} f_0) = 0$ is not a necessary condition for the pair to deform in all directions of the moduli space containing f_0, but the situation could be very subtle if f_0 is a smooth hypersurface and C_0 is a curve, especially in the case where f_0 has a low dimension and C_0 has a low genus. This converse of the Kodaira's theorem reveals subtle differences in deformation theory of the pair of hypersurfaces and their subvarieties. We are interested in the geometric difference between the existence of the first order deformation of the pair $C_0 \subset f_0$ and the existence of the "full" deformation of the pair. This paper is just the first step in this attempt, in which we prove theorem 1.2 below. It gives a necessary condition (i.e. an obstruction) for C_0 to deform to "other hypersurfaces" in the FIRST order. This condition in theorem 1.2 below is different from that in [9]. The main application of it is the proof of the following,

THEOREM 1.1. *There are no smooth elliptic curves in generic hypersurfaces of degree*

$$h \geq 2n-1$$

in the projective space $\mathbf{P}^n$.

Remark This theorem improves Clemens"s bound in [2] by 1. But this is only a weaker version of proposition 5.1 below. Because the result in the theorem is noticeable and requires no buildups in definitions, we state it as the first theorem. The simple bound in the theorem involves many complex issues which may indicate the geometric difference in the deformation theory of a pair of projective varieties. See the remark after proposition 5.1.

The first order deformation of a pair of varieties were rigorously defined and studied by Roy Smith and Robert Varley in [7]. Their main interest lies in a pair of a smooth variety and its divisor, and a sufficient condition for the pair to deform in the first order. Even though it is only for a divisor, it is still a very important view and valid in many extensions. But we are going to bypass it in this paper because we concentrate on a different situation.

Setting for theorem 1.2.

We need to give a formal description on the first order deformation of the pair. See [7] for a more general hypercohomological approach. Let $H^0(\mathcal{O}_{\mathbf{P}^n}(h))$ denote the vector space of homogeneous polynomials of degree h in $n+1$ variables. We use the same letter $f_0 \in H^0(\mathcal{O}_{\mathbf{P}^n}(h))$ to denote the hypersurface $div(f_0) \subset \mathbf{P}^n$, homogeneous polynomial f_0, and its projectivization in $\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$. Let $S \subset \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$ be a subvariety containing f_0 which is a smooth point of S. Also assume that f_0 is a smooth hypersurface. Let

$$X_S \subset S \times \mathbf{P}^n, \tag{1.1}$$

$$X_S = \{(f,x) : f \in S, f(x)=0\}. \tag{1.2}$$

be the universal hypersurface.

Let C be a smooth projective curve of genus g, and

$$c_0 : C \to f_0 \subset P^n$$

a smooth imbedding of C to f_0. Then

$$\bar{c}_0 : C \to \{f_0\} \times f_0 \subset X_S$$

is the induced imbedding. The projection

$$P_S : X_S \to S$$

induces a map on the sections of bundles over C,

$$P_S^s : H^0(\bar{c}_0^*(TX_S)) \to T_{f_0}S, \tag{1.3}$$

where $T_{[f_0]}S \simeq H^0(T_{[f_0]}S \otimes \mathcal{O}_C)$ is the space of global sections of the trivial bundle whose each fibre is $T_{f_0}S$. A pre-image of P_S^s represents a first order deformation of the pair.

In this paper we consider two specific parameter spaces for S:

Assumption (1) The first subvariety S under consideration is the collection of hypersurfaces in the following form:

$$f_0 + \sum_{i=0}^{h} a_i L_0 \cdots \hat{L}_i \cdots L_h, \quad (\hat{L}_i \textit{ is omitted})$$

where $L_i \in H^0(\mathcal{O}_{\mathbf{P}^n}(1)), i = 0, \cdots, h$ are fixed sections whose zeros are distinct, i.e.

$$div(L_i) \neq div(L_j), i \neq j. \tag{1.4}$$

Let

$$A' = \mathbf{C}^{h+1} = \{(a_0, \cdots, a_h)\}$$

be the parameter space of the family. Let $A \subset A'$ that parametrizes smooth hypersurfaces. So $S = A$ in this case.

Assumption (2) Secondly S is the entire space $\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$. We will denote $\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$ by E. So $S = E$ in this case.

The exact formula for the genus of the curve in the hypersurface will depend on the hypersurface. It is not surprised to see the genus is a semi-continuous function on the space of hypersurfaces. Thus our genus formula will involve the space A. Let's first introduce the term reflecting this dependence. Continuing with the notations for the assumption (1), let

$$u_i = L_0 \frac{\partial}{\partial a_0} - L_i \frac{\partial}{\partial a_i}, i = 1, \cdots, h \tag{1.5}$$

be sections of $TA \otimes \mathcal{O}_{\mathbf{P}^n}(1)$. It is easy to see u_i annihilate the universal polynomial F,

$$F(a, x) = f_0(x) + \sum_{i=0}^{h} a_i L_0(x) \cdots \hat{L}_i(x) \cdots L_h(x), \quad (\hat{L}_i \textit{ is omitted}).$$

Hence u_i are tangent to X_A at all points of X_A. Let $G(1)$ be the sub-sheave generated by u_i. We have an imbedding map of sheaves,

$$\bar{c}_0^*(G(1)) \quad \to \quad \bar{c}_0^*(TX_A(1)).$$

Let ϕ_3 be the induced map on their H^1 groups,

$$H^1(\bar{c}_0^*(G(1))) \quad \xrightarrow{\phi_3} \quad H^1(\bar{c}_0^*(TX_A(1)))$$

We'll use the notations: $h^i(E)$ denotes the dimension of $H^i(E)$ for any sheaf E. For any linear map α, $Im(\alpha)$ denotes the image of the map α. Let $N_{c_0}V$ denote the pull-back of any subbundle V of $T\mathbf{P}^n|_{C_0}$ to C. Let $\mathcal{L} = c_0^*(\mathcal{O}_{\mathbf{P}^n}(1))$. Note $d = deg(\mathcal{L})$.

THEOREM 1.2. *Let f_0, C_0, A be as above. Let*

$$\{L_i = 0\} \cap \{L_j = 0\} \cap C_0 = \emptyset, \quad i \neq j. \tag{1.6}$$

Also assume P^s_A is surjective. Then

$$\begin{array}{rrc} & \sigma(c_0, f_0, A) := & (h-2n)d + (n-1)(g-1) \\ (1.7) & & +h^0(c_0^\ast(Tf_0(1)) + (h+1)h^1(\mathcal{L}) \\ & & -dim(Im(\phi_3)) - h^1(c_0^\ast(T\mathbf{P}^n(1)) = 0, \end{array}$$

In this formula, the most difficult term is $h^0(c_0^\ast(Tf_0(1))$, but the most intriguing term is $dim(Im(\phi_3))$. To understand $dim(Im(\phi_3))$ better, we introduce two more terms m, k. Let m be the dimension of the image B of the following composition map $\phi_6 \circ \phi_0$,

$$(1.8) \qquad H^0(\bar{c}_0^\ast(G(1))) \stackrel{\phi_0}{\rightarrow} H^0(\bar{c}_0^\ast(TX_A(1))) \stackrel{\phi_6}{\rightarrow} H^0(N_{c_0}f_0(1)))$$

where ϕ_6 is induced from the composition map of bundles over C,

$$\bar{c}_0^\ast(TX_A(1)) \stackrel{\pi}{\rightarrow} c_0^\ast(Tf_0(1)) \rightarrow N_{c_0}f_0(1),$$

where the projection map π is an important map that is induced from the splitting in the formula (3.3) below, and the π exists only because C_0 can deform to all the other hypersurfaces in A in the first order, i.e. P^s_A is surjective. So

$$B = Im(\phi_6 \circ \phi_0).$$

Now consider the map

$$(1.9) \qquad H^1(\oplus_m \mathcal{O}_C) \stackrel{\phi_5}{\rightarrow} H^1(N_{c_0}f_0(1)))$$

where ϕ_5 is induced from the bundle map of the trivial bundle to the normal bundle $c_0^\ast(N_{c_0}f_0(1))$,

$$\oplus_m \mathcal{O}_C \simeq B \otimes \mathcal{O}_C \rightarrow c_0^\ast(N_{c_0}f_0(1)).$$

Let $k = dim(ker(\phi_5))$. Note that B is unique up to an isomorphism but k, m are uniquely determined by the sections $L_0, \cdots, L_h$ and the first order deformations of C_0 to hypersurfaces collected in A.

Applying theorem 1.2 to smooth curves in a smooth hypersurfaces, we obtain that

Corollary 1.3. *(*Genus formula)

Continuing from theorem 1.2 (with the same assumptions), let g be the genus of a smooth curve C_0 in a general hypersurface of degree h in $\mathbf{P}^n$. In addition, we assume $d > 4(g-1)$. Then

$$g = \frac{(h-2n+1)d + h^0(N_{C_0}f_0(1)) - n + 4 + k}{m - n + 4}.$$

Remark.

(1) Theorem 1.2 proves that if $\sigma(c_0, f_0, A) \neq 0$, then C_0 can't deform to all the hypersurfaces in A in the first order. Thus $\sigma(c_0, f_0, A) \neq 0$ gives us an obstruction

to the deformations of C_0 to other hypersurfaces. However the name, "obstruction" may be misleading because $\sigma(c_0, f_0, A)$ depends on A and ϕ_3, i.e. depends how C_0 deforms to other hypersurfaces in A.

(2) There are lots of work on the bound of genus of the subvariety of a generic hypersurface or a generic complete intersection. We don't mean to include a complete list of results in this area. We only mention those that have a direct relation with our results. Corollary 1.3 has some overlap with the results of Clemens in [2], where he showed that a lower bound of genus is

$$\frac{1}{2}(h-2n+1)d+1.$$

But his bound is not sharp (see [8]), it implies that there are no immersed elliptic curves of in a generic hypersurface of degree

$$h \geq 2n.$$

In section 2 below, we describe and prove a theorem of H. Clemens' on the deformation of hypersurfaces. This is the starting point for the entire paper. We include it here in its completeness because it is not published and we need to use it in this paper. In section 3, we study the deformation of the curve C_0 with the deformation of hypersurface to derive a necessary condition of the pair to deform in the first order. In section 4, we apply the result from section 3 to obtain a genus formula for a smooth curve on a smooth hypersurface (no need to be generic). In section 5, we apply the genus formula to obtain the bounds of hypersurfaces that imply theorem 1.1.

2. Deformation of the hypersurface. The main idea of the proof is to transform the problems of $T\mathbf{P}^n$ to similar types of problems of some isomorphic bundle $\frac{TX_A(1)}{G(1)}$. The exact sequence from this quotient, in return, gives a way to the study of $T\mathbf{P}^n$. The existence of the first order deformations of the pair C_0, f_0 allows us to derive properties in this exact sequence. Thus the isomorphism between $T\mathbf{P}^n$ and $\frac{TX_A(1)}{G(1)}$ serves as an important bridge between two different realms. In this section, we introduce the construction of the vector bundle $\frac{TX_A(1)}{G(1)}$, provided and proved by Herb Clemens ([1]). The curve C_0 is not involved.

Recall $L_0, \cdots, L_h \in H^0(\mathcal{O}_{\mathbf{P}^n}(1))$ satisfy the formulas (1.4) as in assumption (1), and

$$F(a_1, \cdots, a_h, x) = f_0(x) + \sum_{i=0}^{h} a_i L_0(x) \cdots \hat{L}_i(x) \cdots L_h(x), \quad (\textit{omit } L_i) \tag{2.1}$$

is the universal polynomial. Thus

$$\{F = 0\} = X_A \subset A \times \mathbf{P}^n.$$

is also the universal hypersurface, which is smooth. Let $W \subset \mathbf{P}^n$ denote the complement of the proper subvariety

$$\cup_{h \geq j > i \geq 0} \{L_i = L_j = 0\}. \tag{2.2}$$

Let

$$X_W = X_A \cap (A \times W) \qquad (2.3)$$
$$f_0^W = f_0 \cap W.$$

Recall

$$u_i = L_0 \frac{\partial}{\partial a_0} - L_i \frac{\partial}{\partial a_i}, i = 1, \cdots, h \qquad (2.4)$$

are sections of $TA \otimes \mathcal{O}_W(1)$. Since u_i annihilate F, they are tangent to X_W. So let

$$G(1) \subset TX_W(1) \qquad (2.5)$$

be the vector bundle of rank h over X_W that is generated by the sections u_i. Note that because of the condition (2.2) on $L_j, j = 0, \cdots, h$, $G(1)$ is a trivial bundle of rank h over X_W.

For any smooth varieties V_1, V_2, let

$$T_{V_1/V_2}$$

denote the relative tangent bundle of V_1 over V_2, i.e. it is the bundle $TV_1 \oplus \{0\}$ over the variety $V_1 \times V_2$.

The following theorem 2.1 is communicated to us by H. Clemens ([1]), who after learning our construction of the section u_i, proved:

Theorem 2.1. *(H. Clemens)*

$$\frac{TX_W(1)}{G(1)} \simeq T_{W/A}(1), \qquad (2.6)$$

where $T_{W/A}(1)$ is restricted to X_W.

Proof. Consider the exact sequence

$$0 \to \frac{TX_W(1)}{G(1)} \to \frac{T(A\times W)(1)}{G(1)} \to \mathcal{D} \to 0. \qquad (2.7)$$

of bundles over X_W, where $\mathcal{D}$ is some quotient bundle over X_W. Easy to see

$$c_1(\mathcal{D}) = c_1(\mathcal{O}_{\mathbf{P}^n}(h+1))|_{X_w}. \qquad (2.8)$$

Let s be a generic section of $\mathcal{O}_{\mathbf{P}^n}(1)$ that does not have common zeros with $L_i, i = 0, \cdots, h$. Let σ be the reduction of $s\frac{\partial}{\partial a_0}$ in $\frac{T(A\times W)(1)}{G(1)}$. Notice the zeros of σ is exactly

$$div(\sigma) = div(sL_1 \cdots L_h). \qquad (2.9)$$

Since $sL_1 \cdots L_h \in H^0(\mathcal{O}_{\mathbf{P}^n}(h+1))$, σ splits the sequence (2.7). If $L_s \subset \frac{T(A\times W)(1)}{G(1)}$ is the line bundle generated by σ,

$$L_s \oplus \frac{TX_W(1)}{G(1)} = \frac{T(A \times W)(1)}{G(1)}, \qquad (2.10)$$

as bundles over X_W. Secondly, we have another exact sequence

$$0 \to T_{W/A}(1) \to \frac{T(A\times W)(1)}{G(1)} \to \mathcal{D}' \to 0. \tag{2.11}$$

of bundles over X_W, where $\mathcal{D}'$ is some quotient bundle over X_W. By the direct calculation (note $G(1)$ is a trivial bundle):

$$c_1(\mathcal{D}') = c_1(c_0^*(T_{A/W}(1))) = (h+1)(c_1(\mathcal{O}_{\mathbf{P}^n}(1)))|_{X_W}$$

As above, σ splits this sequence (2.11). Hence

$$L_s \oplus T_{W/A}(1) = \frac{T(A\times W)(1)}{G(1)}. \tag{2.12}$$

Comparing the formulas (2.10), (2.12), we obtain

$$\frac{TX_W(1)}{G(1)} \simeq T_{W/A}(1), \tag{2.13}$$

over X_W. $\square$

3. Deformations of curves to other hypersurfaces. In this section, we prove theorem 1.2 and a formula for $dim(Im(\phi_3))$.

Proof. of theorem 1.2: In theorem 1.2, sections $L_1, \cdots, L_h$ satisfy both conditions in formulas (1.4) and (1.6). Consider the exact sequence of Clemens' quotient $\frac{TX_W(1)}{G(1)}$,

$$0 \to \bar{c}_0^*(G(1)) \to \bar{c}_0^*(TX_W(1)) \to \bar{c}_0^*(\tfrac{TX_W(1)}{G(1)}) \to 0$$

This induces the long exact sequence

$$\begin{array}{ccccc} H^0(\bar{c}_0^*(TX_A(1))) & \stackrel{\phi_1}{\to} & H^0(\bar{c}_0^*(\frac{TX_W(1)}{G(1)})) & & \\ & & \downarrow\phi_2 & & \\ & & H^1(\bar{c}_0^*(G(1))) & & \\ & & \downarrow\phi_3 & & \\ & & H^1(\bar{c}_0^*(TX_A(1))) & \stackrel{\phi_4}{\to} & H^1(\bar{c}_0^*(\frac{TX_W(1)}{G(1)})) \to 0. \end{array} \tag{3.1}$$

This exact sequence and theorem (2.1) which says

$$c_0^*(\frac{TX_A(1)}{G(1)}) \simeq c_0^*(T\mathbf{P}^n(1)),$$

yield

$$dim(Im(\phi_3)) + h^1(c_0^*(T\mathbf{P}^n(1)) - h^1(c_0^*(TX_A(1))) = 0. \tag{3.2}$$

Next we calculate $h^1(c_0^*(TX_A(1)))$. Since P_A^s is surjective, we obtain

$$c_0^*(TX_A(1)) \simeq (\oplus_{h+1}\mathcal{L}) \oplus c_0^*(Tf_0(1)), \tag{3.3}$$

where each copy $\mathcal{L}$ is $\mathcal{L} \simeq \mathcal{O}_C \otimes \mathcal{L}$, and the trivial bundle $\mathcal{O}_C$ is generated by the section

$$\frac{\partial}{\partial a_k} - \beta_k, k = 0, \cdots, h,$$

where a_k are affine coordinates of A' defined in the assumption (1). We choose one $\beta_k \in c_0^*(T\mathbf{P}^n)$ for each $\frac{\partial}{\partial a_k}$. Then

$$h^1(\bar{c}_0^*(TX_A(1))) = (h+1)h^1(\mathcal{L}) + h^1(c_0^*(Tf_0(1))) \tag{3.4}$$

Using Riemann-Roch, we obtain that

$$\begin{aligned}
&h^1(c_0^*(Tf_0(1))) \\
&= h^0(c_0^*(Tf_0(1))) - \left(Ch(c_0^*(f_0(1))) \cdot Tod(TC)\right) \\
&= h^0(c_0^*(Tf_0(1))) - \left(c_1(c_0^*(Tf_0(1))) + \frac{n-1}{2}(TC)\right) \\
&= h^0(c_0^*(Tf_0(1))) - \left(c_1(c_0^*(T\mathbf{P}^n(1))) - (h+1)d + \frac{n-1}{2}c_1(TC)\right) \\
&= h^0(c_0^*(Tf_0(1))) + (h-2n)d + (n-1)(g-1)
\end{aligned} \tag{3.5}$$

Combining formulas (3.2), (3.4) and (3.5), we proved theorem 1.2. □

Lemma 3.1. *Assume P_A^s is surjective and $d > 4(g-1)$. Then*

$$dim(Im(\phi_3)) = mg - k,$$

where $k = dim(ker(\phi_5))$ (see formula (1.9)).

Proof. Because P_A^s is surjective, we have the decomposition as in formula (3.3),

$$c_0^*(TX_A(1)) \simeq \oplus_{h+1}\mathcal{L} \oplus c_0^*(Tf_0(1)).$$

Then

$$H^1(c_0^*(TX_A(1)) \simeq \oplus_{h+1}H^1(\mathcal{L}) \oplus H^1(c_0^*(Tf_0(1))).$$

Since $d > 2(g-1)$, $H^1(\mathcal{L}) = 0$. Thus

$$H^1(c_0^*(TX_A(1)) \simeq H^1(c_0^*(Tf_0(1))).$$

Notice that

$$m = dim(B),$$

is the dimension of the image of $\phi_6 \circ \phi_0$, i.e.,

$$B = span(\{c_0^*(L_0\beta_0 - L_k\beta_k)\}_{all\ k}) \subset H^0(c_0^*(N_{c_0}f_0(1))),$$

where $c_0^*(L_0\beta_0 - L_k\beta_k)$ are reduced to $H^0(N_{c_0}f_0(1))$. Now consider the commutative diagram

$$\begin{array}{ccccc}
H^1(\oplus_m\mathcal{O}_C) & \stackrel{\phi_7}{\to} & H^1(c_0^*(Tf_0(1))) \simeq H^1(c_0^*(TX_A(1)) & \stackrel{\phi_3}{\leftarrow} & H^1(c_0^*(G(1))) \\
\downarrow\phi_5 & & \downarrow P & & \\
H^1(c_0^*(N_{c_0}f_0(1))) & = & H^1(c_0^*(N_{c_0}f_0(1))) & &
\end{array}$$

where ϕ_7 is induced from the bundle map of the trivial bundle

$$B \otimes \mathcal{O}_C \simeq \oplus_m\mathcal{O}_C$$

over C to $c_0^*(Tf_0(1))$.[1] There is an exact sequence for the second vertical map P,

$$H^1(TC(1)) \quad \to \quad H^1(c_0^*(Tf_0(1))) \quad \overset{P}{\to} \quad H^1(c_0^*(N_{c_0}f_0(1))). \tag{3.6}$$

Because $d > 4(g-1)$ in both cases where $g = 0$ and $g \neq 0$, $H^1(TC(1)) = 0$. Thus P is injective. Then we obtain

$$dim(Im(\phi_7)) = mg - dim(ker(\phi_5)) = mg - k. \tag{3.7}$$

Then it suffices to prove that

$$Im(\phi_3) = Im(\phi_7).$$

Let $\{U_j\}$ be an affine open covering of C. Let

$$\{\epsilon_k^{j_1 j_2}\}, k = 1, \cdots, h$$

be the representative of an element in the Čech-cohomology for

$$H^1(c_0^*(G(1))) \simeq H^1(\oplus_h \mathcal{O}_C).$$

By the definition of ϕ_3, the image of ϕ_3 is just the co-cycle

$$\{\sum_k \epsilon_k^{j_1 j_2}(c_0^*(L_0\beta_0 - L_k\beta_k))|_{U_{j_1}\cap U_{j_2}}\} \in H^1(c_0^*(Tf_0(1)),$$

where $c_0^*(L_0\beta_0 - L_k\beta_k)$ is regarded as a section in $H^0(c_0^*(Tf_0(1))$ (without modular TC_0 as for B). Notice $H^1(TC(1)) = 0$. Then we have the decomposition

$$H^0(c_0^*(Tf_0(1))) \simeq H^0(TC(1)) \oplus H^0(c_0^*(N_{c_0}f_0(1))).$$

This decomposition shows that any cycle in $H^0(c_0^*(N_{c_0}f_0(1)))$ could have a representative in $H^0(c_0^*(Tf_0(1)))$. Then it suffices to show that the co-cycle

$$\alpha = \{\sum_k \epsilon_k^{j_1 j_2}(c_0^*(L_0\beta_0 - L_k\beta_k))|_{U_{j_1}\cap U_{j_2}}\}$$

is zero if the global sections $L_0\beta_0 - L_k\beta_k$ are tangent to C_0. This is indeed true because P is injective. More specifically, if all $L_0\beta_0 - L_k\beta_k$ are tangent to C_0,

$$\alpha \in Image(H^1(TC \otimes \mathcal{L})) \subset H^1(c_0^*(Tf_0(1))).$$

Again because $d > 4(g-1)$, $H^1(TC \otimes \mathcal{L}) = 0$. Thus $\alpha = 0$. □

4. Genus formula for smooth curves in smooth hypersurfaces in $\mathbf{P}^n$. In this section we apply theorem 1.2 to study the genus of curves C_0 in a smooth hypersurface f_0 in $\mathbf{P}^n$.

We are going to prove corollary 1.3:

Proof. of corollary 1.3 :

Because $deg(\mathcal{L}^* \otimes K) = -d + 2g - 2 < 0$, by the Serre-duality,

$$h^1(\mathcal{L}) = h^0(\mathcal{L}^* \otimes K) = 0.$$

[1] ϕ_7 is well-defined, because $B \otimes \mathcal{O}_C$ is a a trivial bundle.

Consider the twisted Euler sequence pulled back to C:

$$0 \quad \to \quad \mathcal{O}_C \otimes c_0^*(\mathcal{O}_{\mathbf{P}^n}(1)) \quad \to \quad (\oplus_{n+1} c_0^*(\mathcal{O}_{\mathbf{P}^n}(2))) \quad \to \quad c_0^*(T\mathbf{P}^n(1)) \quad \to \quad 0.$$

Then we have the exact sequence on cohomologies

$$H^1(\oplus_{n+1} c_0^*(\mathcal{O}_{\mathbf{P}^n}(2))) \quad \to \quad H^1(c_0^*(T\mathbf{P}^n(1))) \quad \to \quad H^2(\mathcal{O}_C \otimes \mathcal{O}_{\mathbf{P}^n}(1)) \tag{4.1}$$

Since $d > g-1$, $H^1(\oplus_{n+1} c_0^*(\mathcal{O}_{\mathbf{P}^n}(2))) = 0$. By the Grothendieck vanishing theorem ([5]), $H^2(\mathcal{O}_C \otimes c_0^*(\mathcal{O}_{\mathbf{P}^n}(1))) = 0$. Thus $H^1(c_0^*(T\mathbf{P}^n(1))) = 0$. It follows from formula (1.7) and lemma 3.1 that

$$(h-2n)d + (n-1)(g-1) + h^0(c_0^*(Tf_0(1)) - mg + k = 0. \tag{4.2}$$

Consider the exact sequence

$$0 \quad \to \quad TC \otimes \mathcal{L} \quad \to \quad c_0^*(Tf_0(1)) \quad \to \quad c_0^*(N_{c_0} f_0(1)) \quad \to \quad 0.$$

It induces

$$0 \quad \to \quad H^0(TC \otimes \mathcal{L}) \quad \to \quad H^0(c_0^*(Tf_0(1))) \quad \to \quad H^0(c_0^*(N_{c_0} f_0(1))) \quad \to \quad 0.$$

Hence

$$h^0(c_0^*(Tf_0(1)) = h^0(c_0^*(N_{c_0} f_0(1))) + d + 3 - 3g. \tag{4.3}$$

Combining formulas (4.2), (4.3), we complete the proof.
□

5. Smooth elliptic curves in a smooth hypersurfaces in $\mathbf{P}^n$. In this section, we turn our attention to elliptic curves.

PROPOSITION 5.1. *Assume f_0 is a smooth hypersurface of degree h in $\mathbf{P}^n$ and C_0 is a smooth elliptic curve in f_0.*

(1) Let A be the parameter space of hypersurfaces containing f_0 as in theorem 1.2. If P_A^s is surjective, then $h \leq 2n-1$.

(2) If P_E^s is surjective, then $h \leq 2n-2$

Theorem 1.1 follows from proposition 5.1, because if there are smooth elliptic curves in generic hypersurfaces f_0, then P_E^s is surjective. Then proposition 5.1, part (2) says $h \leq 2n-2$. This is the same assertion as that in theorem 1.1.

Remark

(1) A Clemens' theorem in [2] implies that there are no smooth elliptic curves in generic hypersurfaces of degree

$$h \geq 2n$$

in the projective space $\mathbf{P}^n$. We use our method, theorem 1.2 to improve Clemens' inequality by 1.

(2) Furthermore, our bound $2n-2$ only requires the first order deformation of the pair $C_0 \subset f_0$. This also means the bound obtained with the condition of "full" deformation of the pair may be sharper than our bound. To obtain a better bound, one may have to use higher order deformations of the pair $C_0 \subset f_0$. This is indeed the case in [8] for rational curves, and in [3] for elliptic curves in sextic 3-folds.

Thus our bound comes from the existence of the abstract first order deformation, while Clemens and Ran's better bound (or Voisin's for rational curves) for $n = 4$ comes from the existence of the "full" deformation of the pair. The different bounds represent the different deformations of the pair.

(3) Our inequality is not under the "full" deformation condition. Thus it is mostly not sharp if the "full" deformation of the pair is assumed. There are examples showing this: in the case of $n = 4$, Clemens and Ran proved that there are no elliptic curves in generic sextic three-folds ([3]) (with the assumption of "full" deformation). But it was conjectured by J. Harris and proved by G. Xu that our bound is sharp for $n = 3$ ([10]) with the assumption of the "full" deformation. Thus we speculate the sharp bound of h is not a polynomial in n under any deformation conditions. This is contrary to the case of rational curves (the sharp upper bound with the "Full" deformation is $2n-3$). Also we should point it out that, in the proof, if we only require the weaker bound $h \geq 2n$, the f_0 only needs to deform to hypersurfaces in A in the first order.

Proof. proof of proposition 5.1: Let C_0 be a smooth elliptic curve in a smooth hypersurface $f_0 \subset \mathbf{P}^n$ in A. By the genus formula in corollary 1.3,

$$g = \frac{(h-2n+1)d + h^0(N_{C_0}f_0(1)) - n + 4 + k}{m - n + 4}.$$

Thus

$$(h-2n+1)d = m - h^0(N_{C_0}f_0(1)) - k.$$

Since $m \leq h^0(N_{C_0}f_0(1))$, $h \leq 2n-1$.

This shows a generic hypersurface f_0 in the family A can't have a smooth elliptic curve if $h \geq 2n$.[2]

To prove part (2), we only need to come up with a contradiction for $h = 2n-1$. From genus formula, we have

$$g = \frac{h^0(N_{c_0}f_0(1)) - n + 4 + k}{m - n + 4}.$$

Because $k \geq 0$, it suffices to prove that

$$h^0(N_{c_0}f_0(1)) > m = dim(B).$$

(This contradicts $g = 1$). Note $B \subset H^0(N_{c_0}f_0(1))$. We would like to construct a section in $H^0(N_{c_0}f_0(1))$ but not in B.

This uses the entire space $\mathbf{P}_E$ of hypersurfaces. Notice m is determined by the sections $L_i \in H^0(\mathcal{O}_{\mathbf{P}^n}(1))$. Let's carefully choose these sections L_i. Since P_E^s is surjective, then for $\alpha \in H^0(\mathcal{O}_{\mathbf{P}^n}(h))$, there is a section $< \alpha > \in H^0(c_0^*(T\mathbf{P}^n))$ such that

$$(\alpha, < \alpha >) \in H^0(\bar{c}_0^*(TX_E)).$$

It is clear that $< \alpha >$ is unique upto a section of $c_0^*(Tf_0)$. First fix a point $q = c_0(t_0) \in C_0$. Let E_q be any fixed hyperplane in $T_q f_0$. Let L_0 be a section in

$$H_q := H^0(\mathcal{O}_{\mathbf{P}^n}(1) \otimes \mathcal{I}_q)$$

[2] If f_0 is generic, this also can be derived from Clemens' result in [2]. But our f_0 is in A and is not generic.

where $\mathcal{I}_x$ is the ideal sheaf of $\{q\} \subset \mathbf{P}^n$. We define

$$S_{E_q} \subset H_q \times \left(H^0(\mathcal{O}_{\mathbf{P}^n}(1))\right)^h \tag{5.1}$$
$$S_{E_q} = \{(L_0, L_1, \cdots, L_h) :< L_0 L_1 \cdots \hat{L}_i \cdots L_h > |_q \in E_q, i \neq 0\}$$

Next we claim

Claim 5.1: *there are E_q and*

$$L_0 \in H_q, L_1, \cdots, L_h \in H^0(\mathcal{O}_{\mathbf{P}^n}(1))$$

such that

$$(L_0, L_1, \cdots, L_h) \in S_{E_q}$$

and $L_0, L_1, \cdots, L_h$ satisfy formula (1.6), i.e.

$$\{L_i = 0\} \cap \{L_j = 0\} \cap C_0 = \emptyset, i \neq j.$$

Let's prove the claim. For the worst, we may assume $H^0(c_0^*(Tf_0)) = 0$.[3] Then $< \alpha >_q$ is a well-defined vector in $T\mathbf{P}^n|_q$ for any point $q \in C_0$. First for generic $L_0 \in H_q$, and generic $J, L \in H^0(\mathcal{O}_{\mathbf{P}^n}(1))$,

$$< L_0 L \cdots L >_q, < L_0 J L \cdots L >_q$$

are linearly independent vectors. Because if it was not true, by the genericity of all sections, $< L_0 J L \cdots L >_q$ would've been zero. Since

$$L^{h-2} \in H^0(\mathcal{O}_{\mathbf{P}^n}(h-2))$$

for all generic L linearly span the entire space

$$H^0(\mathcal{O}_{\mathbf{P}^n}(h-2)).$$

Thus by the linearity again, we see that for any

$$\alpha \in H^0(\mathcal{O}_{\mathbf{P}^n}(h)),$$

$< \alpha >$ would've been zero at the points of C_0 where $< \alpha >$ lies in Tf_0. This is not true because by the $GL(n+1)$ action on $\mathbf{P}_E \times \mathbf{P}^n$, for any $(n+1) \times (n+1)$ matrix g with the trace being zero, $(-gf_0, gq)$ lies in $TX|_q$ (this is the infinitesimal action).

Now we can choose a subspace $E_q \subset Tf_0|_q$ of dimension $n-2$ such that $< L_0 L \cdots L >_q$ lies in E_q but $< L_0 J L \cdots L >_q$ does not. Also choose such L_0, L that they do not vanish simultaneously at any point of C_0. Next we would like to prove that S_{E_q} is smooth at $(L_0, L, \cdots, L)$. To show this, we consider the analytic subset $U_i = \{L + x_i J\}$ in each $H^0(\mathcal{O}_{\mathbf{P}^n}(1)$, where x_i are complex numbers. Let $L_0' \in H_q$ be another generic section, and $U_0 = \{L_0 + x_0 L_0'\}$ such that

$$< L_0' L \cdots L >_q \notin E_q.$$

[3]If $H^0(c_0^*(Tf_0)) \neq 0$, we should fix a decomposition of the linear space

$$H^0(c_0^*(T\mathbf{P}^n)) = H^0(c_0^*(Tf_0)) \oplus V.$$

Then take $< \alpha >$ to be the V-component of the inverse image of α in the decomposition. Such $< \alpha >$ is unique.

Then

$$S_{E_q}\cap (U_0\times U_1\times U_2\cdots\times U_h),$$

is a subset of $\mathbf{C}^{h+1}$ parametrized by $\{(x_0,\cdots,x_h)\}$, that is defined by

$$g(x_0,\cdots,\hat{x}_i,\cdots,x_h)=0, i=1,\cdots,h. \tag{5.2}$$

for some multi-linear polynomial g in h variables. Let

$$a'=\frac{< L_0'L\cdots L >}{E_a}\in\frac{Tf_q}{E_q}\simeq \mathbf{C},$$

and

$$a=\frac{< L_0JL\cdots L >}{E_a}\in\frac{Tf_q}{E_q}\simeq \mathbf{C}.$$

By our choice, $a\neq 0$ and $a'\neq 0$. The Jacobian matrix of the formula (5.2) at the origin(corresponding to $(L_0L\cdots L)$) is

$$\begin{pmatrix} a' & a & a & \cdots & a & 0\\ a' & 0 & a & \cdots & a & a\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ a' & a & \cdots & a & 0 & a\end{pmatrix} \tag{5.3}$$

This matrix is row-reduced to

$$\begin{pmatrix} a' & a & 0 & \cdots & a & 0\\ 0 & -a & 0 & \cdots & 0 & a\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & -a & a\end{pmatrix} \tag{5.4}$$

which has the full rank. This shows S_{E_q} is smooth at $(L_0,L,\cdots,L)$. Next we shrink S_{E_q} around $(L_0,L,\cdots,L)$ to make it irreducible. That is to let

$$S_{E_q,\epsilon}=S_{E_q}\cap (U_0^\epsilon\times U^\epsilon\cdots U^\epsilon)$$

for sufficiently small $\epsilon\in\mathbf{C}$, where U^ϵ is the open disk of $H^0(\mathcal{O}_{\mathbf{P}^n}(1)$ centered at $(L_0,L\cdots,L)$ with radius ϵ and

$$U_0^\epsilon=\{L_0+x_0L_0' : |x_0|\leq\epsilon\}.$$

It is clear that $S_{E_q,\epsilon}$ is symmetric under the permutations of $L_1,\cdots,L_h$. For any $L_1,L_2\in U^\epsilon$ which have no common zeros along C_0, by the dimension count and similar infinitesimal argument as in the formula (5.3), we can find other sections $L_0\in H_q$ and $L_3,\cdots,L_h$ such that $(L_0,L_1,L_2,\cdots,L_h)\in S_{E_q,\epsilon}$, i.e. the projection of $S_{E_q,\epsilon}$ to the second and third components,

$$U^\epsilon\times U^\epsilon$$

is surjective. Since $S_{E_q,\epsilon}$ is irreducible (because it is smooth) and symmetric, we proved that for the generic point $(L_0,L_1,\cdots,L_h)\in S_{E_q,\epsilon}$, $L_i,L_j, i\neq j, i\neq 1\neq j$ do

not have common zeros along C_0. Also L_0 does not have common zeros with any of other L_i along C_0 because L_0 does not have common zeros with L along C_0 for the center $(L_0, L, \cdots, L) \in S_{E_q,\epsilon}$. This proves the claim 5.1. Let $L_0, L_1, \cdots, L_h$ satisfy the claim 5.1. Also let

$$L_0 < L_1 \cdots L_h > -L_k < L_0 L_1 \cdots \hat{L}_k \cdots L_h > \in H^0(c_0^*(Tf_0(1))),$$

lie in E_q at q for all $k \neq 0$ (where $\hat{}$ means "omitting"). Then we apply the sections $L_0, L_1, \cdots, L_h$ to construct the subspace A as in section 1. We obtain the integer m which is the dimension of corresponding B. Then each section $\beta \in B$ must lie in E_q at q. Next we construct a section of $H^0(c_0^*(N_{c_0} f_0(1))$ not in B. By the $GL(n+1)$ action on $\mathbf{P}_E \times \mathbf{P}^n$, there are sections $L_1', \cdots, L_{h-1}'$ such that

$$< L_0 L_1' \cdots L_{h-1}' > \in H^0(c_0^*(T\mathbf{P}^n))$$

does not lie in E_q at q, where $L_0 \in H_q$. Let $L_h' \in H^0(c_0^*(T\mathbf{P}^1))$ be any section not in H_q. This shows that

$$L_0 < L_1' \cdots L_h' > -L_h' < L_0 L_1' \cdots L_{h-1}' >$$

is reduced to a non-zero section in $H^0(c_0^*(N_{c_0} f_0(1))$, but it is not in

$$B \subset H^0(c_0^*(N_{c_0} f_0(1))),$$

because it does not lie in E_q at q. Thus

$$dim(H^0(c_0^*(N_{c_0} f_0(1)))) > m = dim(B).$$

We complete the proof.

$\square$

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